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DYNAMIC BEHAVIOUR OF CYLINDRICAL SHELLS STRENGTHENED WITH RING RIBS—PART I. INFINITELY LONG SHELL

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Abstract—The paper considers the dynamic behaviour of infinitely long cylindrical shells made of ideally rigid-plastic material and strengthened with ring ribs of limited rigidity. The principal assumptions are identical with those made in the work [1] where shells with absolutely rigid rings were considered. Distribution of moments and displacements is determined. The values of residual displacements are found as well as of motion stopping times for various loads and parameters of the shell and its strengthening ribs. The maximal residual displacement accumulated in the shell for the entire period of motion under the conditions of "local" collapse is shown to be practically independent of the ring rigidity and to coincide rather accurately with the maximal residual displacement of the shell having absolutely rigid ribs. It is shown that all the results obtained can be applied, after suitable redefinitions, to the case of shells strengthened in the span between the supporting rings, by longitudinal and ring-like ribs of "small" rigidity. The results obtained are also valid for cylindrical shells of finite length with moving supports.

NOTATION

$2l, R, 2h$	the length of a span, the radius of the middle surface, the thickness of a shell wall, respectively
T_2	the turning force per unit of length carried by a facing
$2\delta, d$	the height and the breadth of a rib, respectively
M_x	the bending moment per unit of the circumference of the middle surface
w	the radial displacement
p	the uniformly distributed pressure
t_0	the duration of pulse action
γ_{01}, γ_{02}	the surface density of the material of a facing and ribs, respectively
x	the coordinate on the generatrix
t	the time
σ_{01}, σ_{02}	the yield limit for stretching-pressing the facing and the ribs, respectively
ν_{01}, ν_{02}	the specific weight of a facing and ribs, respectively.

INTRODUCTION

THE dynamic behaviour of cylindrical shells strengthened with rings from ideal rigid-plastic material and under the action of impulsive loads was studied by Hodge [1–3], where infinitely long shells were considered and the rings were assumed to be absolutely rigid. Thus, the behaviour of a smooth shell with the length equal to the distance between the rings was effectively studied in [1–3]. The behaviour of real shells strengthened with rings of limited rigidity has a number of peculiarities which are considered in the present paper. Just as in [1] it is assumed that the material of the shell and rings is ideally rigid-plastic and satisfies a simplified yield condition (Fig. 1) and the flow law associated with it. An impulse of rectangular form (Fig. 2) is applied. The duration of its action is assumed to be sufficiently short for the deformations to be regarded as small compared with unity.

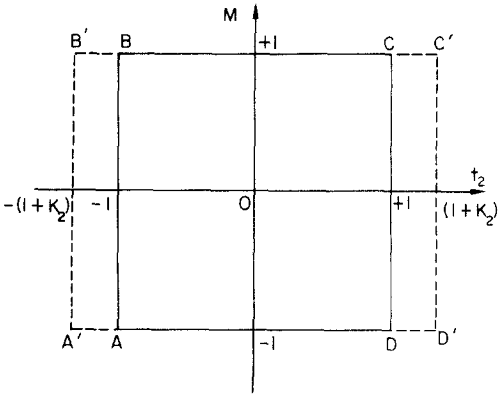

FIG. 1.

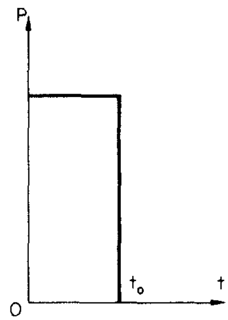

FIG. 2.

To simplify the study it is assumed that the strengthening elements have a rectangular cross-section and their torsional rigidity is neglected. Under the above assumptions we study the character of shell motions for various values of impulsive loads, various stiffnesses of strengthening rings and various parameters of shell spans. It is shown that all the results hitherto obtained by means of corresponding redefinitions are applicable in the case of shells strengthened in the span between stiff longitudinal ribs and/or ring ribs of "small" stiffness. The results obtained are also valid for cylindrical shells of finite length with moving supports. It should be noted that at the "local" collapse the maximal residual displacement accumulated in the shell for the time of its motion practically does not depend on ring stiffness and with a good accuracy coincides with maximal residual displacement in the shell with absolute rigid rings.

1. AXISYMMETRICAL SQUEEZING

Just as in [1–3] let us consider first an infinitely long shell strengthened with equidistant rings, the distance being equal to $2l$. Then under the action of equally distributed impulsive load P the behaviour of all the spans in the shell is identical, so it is sufficient to study the behaviour of one span. Then the equation of the cylindrical shell motion in the span between the rings in dimensionless form is the same as in [1]:

$$m'' + 2\mu^2(t_2 + P - \ddot{W}) = 0 \tag{1.1}$$

and the equation of ring motion has the form

$$p + q^- + q^+ - (1 + a_0) - \lambda \ddot{W}_0 = 0 \tag{1.2}$$

where

$$m = M_x/\sigma_{01}h^2,\ t_2 = T_2/2\sigma_{01}h,\ P = pR/2\sigma_{01}h,\ \mu = l/\sqrt{(Rh)}$$
$$w = w\gamma_{01}R/2\sigma_{01}ht_0^2,\ a_0 = \delta\sigma_{02}/h\sigma_{01},\ q^{\pm} = m'(0,\tau)/2\mu\theta$$
$$\theta = d/\sqrt{(Rh)},\ \lambda = 1 + \gamma_{02}/\gamma_{01},\ W_0(\tau) = W(0,\tau).$$

The primes indicate differentiation with respect to the dimensionless coordinate $y = x/l$ and the dots with respect to dimensionless time $\tau = t/t_0$. The origin of the coordinates is chosen at a supporting ring.

Under the action of relatively low loads and at sufficiently great rigidity of rings there will be no shell motion until the load reaches its maximum value of P_M for the span [1]:

$$P_M = 1 + 2/\mu^2. \tag{1.3}$$

In this case the shearing forces on the bay support are equal to

$$q^{\pm} = \frac{m'(0,\tau)}{2\mu\theta} = \frac{2}{\mu\theta}. \tag{1.4}$$

If at the given value of P_M the stress in the ring reaches the yield stress σ_{02}, then substituting (1.4) into (1.2) at $W_0(\tau) \equiv 0$, we obtain the equation

$$\mu^2\theta a_0 - 4\mu - 2\theta = 0.$$

The root of this equation

$$\mu_* = 2[1 + (1 + \theta^2 a_0/2)^{\frac{1}{2}}]/\theta a_0 \tag{1.5}$$

defines the optimal placing of rings at which for the loads close to the static maximum ones "the local" collapse of the shell in the span between the rings ($\mu \geq \mu_*$) is replaced by the "general" collapse of the shell, together with the rings which corresponds to axisymmetric squeezing ($\mu < \mu_*$). If $\mu < \mu_*$ and $P \leq P^0$

$$P^0 = 1 + \theta a_0/(\theta + 2\mu) \tag{1.6}$$

then there is no shell motion. If $\mu < \mu_*$, but $P > P^0$, then there is shell motion which corresponds to its axisymmetric squeezing. In this case $W(y,\tau) = W_0(\tau)$ and $t_2 = -1$. Substituting these values into equation (1.1) and integrating it under the conditions

$$m'(1,\tau) = 0, \qquad m(0,\tau) = m_0(\tau) \ldots (-1 \leq m_0 \leq 1)$$

we obtain

$$m = m_0 + \mu^2(\ddot{W}_0 + 1 - P)(y-2)y.$$

Then from (1.2) we have

$$\ddot{W}_0 = [(P-1)(2\mu + \theta) - \theta a_0]/(2\mu + \lambda\theta).$$

Taking into consideration the initial conditions

$$W_0(0) = \dot{W}_0(0) = 0$$

we obtain

$$\begin{gathered} m = \frac{m_0+\alpha\theta\mu^2(2-y)y}{(2\mu+\lambda\theta)}, \qquad t_2 = -1, \\ W_0(\tau) = [(P-1)(2\mu+\theta)-\theta a_0]\tau^2/2(2\mu+\lambda\theta) \\ \alpha = (P-1)(\lambda-1)+a_0 \ldots (0 \le y \le 1, 0 \le \tau \le 1). \end{gathered} \tag{1.7}$$

From (1.7) it is evident that $\dot{W}_0(\tau) \ge 0$ at $P \ge P^0$. As for the axisymmetric squeezing we should have $-1 \le m \le 1$, then the solution (1.7) is valid under the condition that

$$P \le P_0 = (P_M - \alpha_1 P_{1*})/(1-\alpha_1)$$
$$\alpha_1 = (\mu+2\theta)/(\mu+2\lambda\theta),\ P_{1*} = 1+2(\mu\theta a_0-3)/\mu(\mu+2\theta)$$

For $\tau \ge 1$ a shell will continue its motion under its own inertia ($P \equiv 0$). The corresponding solution of equations (1.1) and (1.2), with the use of the initial conditions for $W_0(\tau)$ from (1.7) at $\tau = 1$, has the form

$$\begin{gathered} m = m_0+\theta\mu^2(1+a_0-\lambda)(2-y)y/(2\mu+\lambda\theta), \qquad t_2 = -1, \\ W_0(\tau) = \{P(2\mu+\theta)(2\tau-1)-[2\mu+\theta(1+a_0)]\tau^2\}/2(2\mu+\lambda\theta) \qquad (0 \le y \le 1, 1 \le \tau \le \tau^0). \end{gathered} \tag{1.8}$$

The solution (1.8) will be valid if it satisfies the inequalities $m'(0,\tau) \ge 0$ and $\dot{W}(y,\tau) \ge 0$, or

$$\psi = \sigma_{01}v_{02}/\sigma_{02}v_{01} \le 1, \qquad \tau \le \tau^0 = P/P^0. \tag{1.9}$$

The residual displacement is

$$W_0(\tau^0) = (2\mu+\theta)P(P-P^0)/2P^0(2\mu+\lambda\theta). \tag{1.10}$$

It should be noted, that in the case of identical material of the shell and rings the bending moment in (1.8) does not depend on y. It is due to the fact, that under the axisymmetric squeezing the behaviour of an infinitely long shell is identical to that of rings made of the same material.

The condition of immobility of rings

Further, if no special stipulations are made, we will consider the shell spans for which $\mu \ge \mu_*$. In this case dynamic loads either do not cause any motion at all or cause a motion with a bending in the span. Then, depending on the rigidity of rings and the value of the acting load the rings either remain motionless or accompany the shell motion.

Further, we shall consider only the second possibility, as the first one corresponds to the motion of a smooth shell clamped at the ends, studied in [1]. For this we should define first the largest loads under which the rings are motionless. From solution [1] for shells with motionless supports it follows that the shearing forces at the supports do not increase with time. Thus, if rings are motionless during the load action they will remain motionless.

From [1] we obtain

$$q^+ = \begin{cases} [\mu^2(P-1)+6]/4\mu\theta \ldots (P_M \le P \le P_{1M}) \\ [6(P-1)]^{\frac{1}{2}}/2\theta \ldots (P \ge P_{1M}) \end{cases} \tag{1.11}$$

where

$$P_{1M} = 1+6/\mu^2. \tag{1.12}$$

Naturally, depending on the rigidity of the rings, the stresses in them may reach yield stress under the loads $P \le P_{1M}$ as well as under the loads $P > P_{1M}$. Consequently, supposing in (1.2) $\dot{W}_0(\tau) \equiv 0$ and substituting (1.11) into it, we shall obtain the following maximal loads under which the rings remain motionless (though the shell in the span is in motion):

$$\begin{aligned} P_{1*} &= 1+2(\mu\theta a_0-3)/\mu(\mu+2\theta) \ldots (\mu_* \le \mu \le \mu_1) \\ P_{2*} &= 1+6/\mu_1^2 \ldots (\mu \ge \mu_1) \end{aligned} \tag{1.13}$$

where

$$\mu_1 = 3[1+(1+2\theta^2 a_0/3)^{\frac{1}{2}}]/\theta a_0 \tag{1.14}$$

is a root of the equation in the expressions obtained from (1.2) by substituting into (1.11), the values $P = P_{1M}$ from (1.12) and $\dot{W}_0(\tau) \equiv 0$.

From (1.13), (1.14) it is obvious that the maximum dynamic load for the rings under fixed values θ and a_0 increases with an increase of μ over the interval

$$\mu_* \le \mu \le \mu_1 \tag{1.15}$$

and does not depend on μ at $\mu > \mu_1$. Such a qualitative difference is the result of two essentially different types of the shell span motion: with a plastic hinge ($\mu \le \mu_1$) and with a plastic zone in the middle of the span ($\mu > \mu_1$).

2. SHELLS WITH "SHORT" SPANS UNDER "MODERATE" LOADS

If the excess of the load P over P_{1*} is not considerable then though the motion of the shell span is accompanied by that of the ring, it is natural to assume that the plastic regime and the corresponding fields of displacement velocities are the same as in case of motionless supports [1]. Thus for the shells with the span (1.16) the regime AB is assumed for which

$$t_2 = -1 \tag{2.1}$$

and the corresponding field of displacement velocities has the form

$$\dot{W}(y,\tau) = \dot{W}_0(\tau)+[\dot{W}_1(\tau)-\dot{W}_0(\tau)]y \ldots (0 \le y \le 1). \tag{2.2}$$

Here the dimensionless displacement $W(y, \tau)$ is defined by formula (1.2) and

$$W_1(\tau) = W(1, \tau).$$

Substituting (2.1) and (2.2) into (1.1) and integrating the obtained equation under the boundary conditions

$$m(0,\tau) = -1, \qquad m(1,\tau) = 1, \qquad m'(1,\tau) = 0 \tag{2.3}$$

we obtain

$$m = \mu^2[(\ddot{W}_1 - \ddot{W}_0)y^3/3 - (P-1-\ddot{W}_0)y^2] + Ay - 1 \tag{2.4}$$

$$\mu^2[2\ddot{W}_1 + \ddot{W}_0 - 3(P-1)] + 6 = 0 \tag{2.5}$$

$$q^{\pm} = A/2\mu\theta \tag{2.6}$$

where

$$A = 2 - \mu^2[(\ddot{W}_1 - \ddot{W}_0)/3 - P + 1 + \ddot{W}_0].$$

Substituting (2.6) into (1.2) and integrating the obtained equation under zero initial conditions we obtain

$$W_0(\tau) = \alpha_1(P - P_{1*})\tau^2/2 \ldots (0 \le \tau \le 1). \tag{2.7}$$

Then from (2.5) we obtain

$$W_1(\tau) = \frac{3\tau^2[(1-\alpha_1/3)(P-P_{1*}) + P_{1*} - P_M]}{4} \qquad 0 \le \tau \le 1. \tag{2.8}$$

Thus the complete solution for the first phase of the motion will be

$$m = \mu^2\{[P_{1*} - P_M + (1-\alpha_1)(P-P_{1*})]y^3/2 - [(1-\alpha_1)P + \alpha_1 P_{1*} - 1]y^2\}$$
$$+ \{2 + [P_M + \alpha_1 P_{1*} + (1-\alpha_1)P - 2]\mu^2/2\}y - 1, \qquad t_2 = -1 \tag{2.9}$$

$$W(y,\tau) = \left\{\frac{3y[(P_{1*} - P_M) + (1-\alpha_1)(P-P_{1*})]}{2} + \alpha_1(P - P_{1*})\right\}\tau^2/2$$
$$(0 \le \tau \le 1, \quad 0 \le y \le 1, \quad 0 < \mu \le \mu_1, \quad P_{1*} \le P \le P_*). \tag{2.10}$$

The necessary limitation ($-1 \le m \le 1$) for the regime AB will be satisfied, when inequality (2.10) is also satisfied, where

$$P_* = (P_{1M} - \alpha_1 P_{1*})/(1-\alpha_1). \tag{2.11}$$

Note that $P_* \ge P_{1M}$, i.e. in the case of rings with limited rigidity, the load, at which the plastic domain occurs, exceeds the corresponding load in the case of absolutely rigid rings.

For $\tau \ge 1$ the load is vanishing ($P \equiv 0$). Then, integrating, as above, the equations of motion (1.1) and (1.2), and considering (2.1), (2.3) and initial conditions (2.7), (2.8) for $\tau = 1$), we obtain the solution

$$\left\{\begin{aligned} m &= \mu^2[(\alpha_1 P_{1*} - P_M)y^3/2 - (\alpha_1 P_{1*} - 1)y^2 + (\alpha_1 P_{1*} + 3P_M - 4)y/2] - 1, \\ t_2 &= 1, \\ W(y,\tau) &= W_0(\tau) + [W_1(\tau) - W_0(\tau)]y \ldots (0 \le y \le 1, 1 \le \tau \le \tau_0) \\ W_0(\tau) &= \alpha_1[(P-P_{1*})(2\tau-1) - P_{1*}(\tau-1)^2]/2 \ldots (1 \le \tau \le \tau_0) \\ W_1(\tau) &= \{[(3-\alpha_1)(P-P_{1*}) + 3(P_{1*} - P_M)](2\tau-1) - (2 + P_{1M} - \alpha_1 P_{1*})(\tau-1)^2\}/4 \\ &\quad \ldots (1 \le \tau \le \tau_0) \end{aligned}\right. \tag{2.12}$$

$$(1 - 6/\mu^2)/\alpha_1 \le P_{1*} \le P \le P_*, \qquad \mu_* \le \mu \le \mu_1. \tag{2.13}$$

In this case the condition $\dot{W}_0(\tau) \geq 0$ leads to the inequality

$$\tau \leq \tau_0 = P/P_{1*}. \tag{2.14}$$

At the moment $\tau = \tau_0$ the ring stops its motion, though the shell in the span is still moving as

$$\dot{W}_1(\tau_0) = 3\tau_0(P_{1*} - P_M)/2 \geq 0 \ldots (\mu \geq \mu_*).$$

Consequently, it is necessary to find a solution under the same regime AB and for $P \equiv 0$, $\dot{W}_0(\tau) \equiv 0$. As initial conditions we use the following

$$W(0, \tau_0) = W_0(\tau_0), \qquad W(1, \tau_0) = W_1(\tau_0), \qquad \dot{W}(1, \tau_0) = \dot{W}_1(\tau_0). \tag{2.15}$$

The corresponding solution is

$$\begin{aligned} m &= -(1+\mu^2/2)y^3 + \mu^2 y^2 + (3-\mu^2/2)y - 1, \quad t_2 = -1, \\ W(y, \tau) &= W_0(\tau_0) + [W_1(\tau) - W_0(\tau_0)]y, \\ W_1(\tau) &= W_1(\tau_0) + \dot{W}_1(\tau_0)(\tau - \tau_0) - 3P_M(\tau - \tau_0)^2/4 \\ &(\tau_0 \leq \tau \leq \tau^0, \quad \mu_* \leq \mu \leq \mu_2, \quad P_{1*} \leq P \leq P_*). \end{aligned} \tag{2.16}$$

In this case the necessary inequalities for the regime AB [1] will be satisfied, if

$$\mu_* \leq \mu \leq \sqrt{6}. \tag{2.17}$$

The condition $\dot{W}_1(\tau) \geq 0$ leads to the inequality

$$\tau \leq \tau^0 = P/P_M. \tag{2.18}$$

At the moment $\tau = \tau^0$ the motion is stopped and the maximal residual displacement is determined by the formula

$$W_1(\tau^0) = W_1(\tau_0) + \dot{W}_1(\tau_0)(\tau^0 - \tau_0) - 3P_M(\tau^0 - \tau_0)^2/4. \tag{2.19}$$

Note that the time of stopping (2.18) for the shells with spans (2.17) with mobile rings coincides with the corresponding time of stopping the motion in case of motionless rings [1]. The solution obtained in the present section is valid for shells with "short" spans

$$\mu_* \leq \mu \leq \mu_2 = \min(\sqrt{6}, \mu_1). \tag{2.20}$$

and under "moderate" loads

$$P_{1*} \leq P \leq P_*. \tag{2.21}$$

If $\mu_1 > \sqrt{6}$ then under the "moderate" loads (2.21) in the shells with "moderate" spans

$$\max(\sqrt{6}, \mu_*) = \mu_3 \leq \mu \leq \mu_1 \tag{2.22}$$

at the moment $\tau = \tau_0$ of ring stopping there occurs near the ring a plastic domain (regime AD). As the corresponding motion takes place with the motionless rings, the solution may be obtained in the same way as in [1]. Then the time of stopping may be determined by the equality

$$\tau^0 = \tau_0\{1 + 3(P_{1*} - P_M)[1 - (P_{1M} - 1)^{\frac{1}{2}}]/2(2 - P_{1M})\} \tag{2.23}$$

and the residual displacement has the form

$$W_1(\tau^0) = W_1(\tau_0) + [3\tau_0(P_{1*} - P_M)/2(2 - P_{1M})]^2 \times [2 - P_{1M} + (P_{1M} - 1)\ln(P_{1M} - 1)]/2 \tag{2.24}$$

where $W_1(\tau_0)$ and τ_0 is derived from expressions (2.12) and (2.14).

3. SHELLS WITH "SHORT" SPANS UNDER "HIGH" LOADS

If $\mu_* \le \sqrt{6}$ and

$$P \ge P_{**} = \begin{cases} P_* \ldots (\mu_* \le \mu \le \mu_1) \\ P_{2*} \ldots (\mu \ge \mu_1) \end{cases} \tag{3.1}$$

then in the shell, two plastic regimes will be realized

$$AB(0 \le y \le y_1) \quad \text{and} \quad B(y_1 \le y \le 1).$$

Therefore, the distribution of the displacement velocities will be as follows

$$\dot{W}(y,\tau) = \dot{W}_0(\tau) + [\dot{W}_{01}(\tau) - \dot{W}_0(\tau)]y/y_1, \; \dot{W}_{01}(\tau) = \dot{W}(y_1,\tau) \ldots (0 \le y \le y_1).$$

Substituting this value into (1.1) and integrating it under the boundary conditions

$$m(0,\tau) = -1, m(y_1 - 0, \tau) = m(y_1 + 0, \tau) = 1, m'(y_1 - 0, \tau) = m'(y_1 + 0, \tau) = 0 \tag{3.2}$$

we obtain

$$\begin{gathered} m = \mu^2[\beta y^3/3 + (\ddot{W}_0 - P + 1)y^2] + cy - 1 \ldots (0 \le y \le y_1) \\ \mu^2 y_1^2[2\beta y_1 + 3(\ddot{W}_0 - P + 1)] + 6 = 0 \end{gathered} \tag{3.3}$$

$$\beta(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}[\dot{W}_{01}(\tau) - \dot{W}_0(\tau)]/y_1, \qquad c = [2 - \mu^2 y_1^2(\beta y_1/3 + \ddot{W}_0 - P + 1)]/y_1.$$

In the interval $y_1 \le y \le 1$ we have

$$m(y,\tau) = 1, \qquad t_2 = -1, \qquad \ddot{W}(y,\tau) = P - 1.$$

Integrating the last equation under the zero initial conditions and taking into consideration the continuity of displacements and velocities on the boundary $y = y_1$

$$W(y_1 - 0, \tau) = W(y_1 + 0, \tau) = W_{01}(\tau), \qquad \dot{W}(y_1 - 0, \tau) = \dot{W}(y_1 + 0, \tau) = \dot{W}_{01}(\tau)$$

we obtain

$$\begin{gathered} \dot{W}(y,\tau) = \dot{W}_{01}(\tau) = (P-1)\tau, \qquad W(y,\tau) = W_{01}(\tau) = (P-1)\tau^2/2 \\ (y_1 \le y \le 1, \qquad 0 \le \tau \le 1). \end{gathered} \tag{3.4}$$

Now, from equation (3.3) for the value $\ddot{W}_0$, by integrating it under zero initial conditions, we obtain

$$\dot{W}_0(\tau) = (P - 1 - 6/\mu^2 y_1^2)\tau. \tag{3.5}$$

From equation (1.2), using expressions (3.3)–(3.5) we determine the unknown boundary

$$\begin{gathered} y_{01} = v_0 B, \qquad v_0 = [6/\mu^2(P-1)]^{\frac{1}{2}} \\ B = [3(P-1)/2]^{\frac{1}{4}} . [1 + (1 + 2\lambda\theta^2\alpha/3)^{\frac{1}{2}}]/\theta\alpha. \end{gathered}$$

Consequently, as in the case of motionless rings, the plastic zone which occurs in the middle of the span under loads (3.1) remains stationary during the load action and its value depends on the "static" (θ, a_0) and "dynamic" (λ) rigidities of rings.

The complete solution for the first phase ($0 \leq \tau \leq 1$) of motion is

$$\begin{gathered} 0 \leq y \leq y_{01} : m = 1-2(1-y/y_{01})3, \qquad t_2 = -1 \\ W(y,\tau) = [P-1-6(y_{01}-y)/\mu^2 y_{01}^3]\tau^2/2 \end{gathered} \tag{3.6}$$

$$\begin{gathered} y_{01} \leq y \leq 1 : m(y,\tau) = 1, \qquad t_2 = -1, \qquad W(y,\tau) = (P-1)\tau^2/2 \\ (0 \leq \tau \leq 1, \qquad P \geq P_{**}, \qquad \mu \geq \mu_*). \end{gathered} \tag{3.7}$$

The necessary inequalities [1] for the considered regimes are satisfied and the condition $\dot{W}(y,\tau) \geq 0$ leads to the condition $y_{01} \geq v_0$ or $B \geq 1$. The latter will be satisfied if $P \geq P_{2*}$. This inequality is satisfied as in the interval $\mu_* \leq \mu \leq \mu_1$ (which is not difficult to check) $P_* \geq P_{2*}$.

Since v_0 determines the size of the plastic domain at the motionless rings, the inequality $y_{01} \geq v_0$ means that a decrease in rings rigidity leads to a decrease in the size of the plastic domain in comparison with that in the shell with absolutely rigid rings under the same load.

For $\tau \geq 1$ the load is vanishing. Therefore, making calculations analogous to the above ones we obtain (for $P \equiv 0$)

$$\left\{\begin{array}{l} 0 \leq y \leq y_1(\tau) : m = 2y_1^3\dot{x}(y/y_1)^3 - 2(1+2\dot{x}y_1^3)(y/y_1)^2 + 2(2+y_1^3\dot{x})y/y_1 - 1, \qquad t_2 = 1, \\ \qquad \dot{W}(y,\tau) = P-\tau-6x(y_1-y)/\mu^2 \\ y_1(\tau) \leq y \leq 1 : m(y,\tau) = 1, \qquad t_2 = 1, \qquad \dot{W}(y,\tau) = P-\tau \end{array}\right. \tag{3.8}$$

$$(1 \leq \tau \leq \tau_{00}, \qquad \mu \geq \mu_*, \qquad P \geq P_{**}, \qquad \psi \leq 1)$$

$$\left\{\begin{array}{l} y_1(\tau) = 3\{1+[1+2\lambda\theta^2\alpha(\tau)/3]^{\frac{1}{2}}\}/\mu\theta\alpha(\tau), \\ \alpha(\tau) = (1-\lambda)(1-P/\tau)+a_0, \qquad x(\tau) = \tau/y_1^3. \end{array}\right. \tag{3.9}$$

Since $y_1(\tau)$ is a monotonically increasing function of time, after removing the load the plastic domain begins to shrink. The conditions $y_1(\tau) \leq 1$ and $\dot{W}(y,\tau) \geq 0$ reduce to the inequalities

$$\tau \leq \tau_{00} = \left\{\begin{array}{l} \tau_1 = P/P_* \ldots (\mu_* \leq \mu \leq \mu_1) \\ \tau_0 = P/P_{2*} \ldots (\mu \geq \mu_1). \end{array}\right. \tag{3.10}$$

In this case τ_0 determines the time of ring stopping when the plastic domain still exists, τ_1 determines the time of the plastic domain vanishing when the ring is still in motion. Indeed,

$$y_1(\tau_1) = 1, \qquad \dot{W}_0(\tau_1) = \tau_1(P_* - P_{1M}) \geq 0 \ldots (\mu_* \leq \mu \leq \mu_1)$$

as $P_* \geq P_{1M}$. For $\mu = \mu_1$ the time of the plastic domain vanishing coincides with that of the ring stopping.

Now we will study the behaviour of the shells with the span $\mu_* \leq \mu \leq \mu_1$. Since at the instant of time $\tau = \tau_1$ the plastic domain shrank into a plastic in the middle of the span and the rings are still in motion, the solution will be the same as in (2.12) only instead of initial conditions at $\tau = 1$, as in (2.12), one should use the corresponding initial condition at $\tau = \tau_1$.

Then the expressions for t_2 and m coincide with (2.12) and for the displacement velocity we obtain

$$\begin{gathered} \dot{W}(y,\tau) = \dot{W}_0(\tau)+[\dot{W}_1(\tau)-\dot{W}_0(\tau)]y, \qquad \dot{W}_0(\tau) = \tau_1(P_*-P_{1M})-\alpha_1 P_{1*}(\tau-\tau_1), \\ \dot{W}_1(\tau) = P-\tau_1-(2+P_{1M}-\alpha_1 P_{1*})(\tau-\tau_1)/2. \end{gathered} \tag{3.11}$$

Here

$$\tau_1 \le \tau \le \tau_0 = P/P_{1*}, \qquad 0 \le y \le 1, \qquad \mu_* \le \mu \le \mu_1, \qquad P \ge P_*.$$

At the moment of time $\tau = \tau_0$ the rings stop, but the shell in the span between the rings is in motion according to

$$\dot{W}_1(\tau_0) = 3\tau_0(P_{1*}-P_M)/2 \ge 0$$

the equality being possible only at $\mu = \mu_*$ $(P_{1*} = P_M)$.

For $\tau \ge \tau_0$ the corresponding solution is obtained in the same way as in (2.16), so that m and t_2 coincide with the expressions (2.16) and for the displacement velocities we obtain

$$\dot{W}(y,\tau) = 3P_M(\tau^0-\tau)y/2\ldots(\tau_0 \le \tau \le \tau^0, 0 \le y \le 1, \mu_* \le \mu \le \mu_2). \tag{3.12}$$

At the moment of time $\tau = \tau^0 = P/P_M$ the motion of the shell is stopped completely.

The maximal residual displacement is expressed by the formula

$$\begin{gathered} W_1(\tau^0) = 3P_M(\tau^0-\tau_0)^2/4+(P-\tau_1)(\tau_0-\tau_1)-(2+P_{1M}-\alpha_1 P_{1*})(\tau_0-\tau_1)^2/4 \\ +[P(P-1)-(P-\tau_1)^2]/2\ldots(\mu_* \le \mu \le \mu_2, P \ge P_*). \end{gathered} \tag{3.13}$$

In the shells with the span $\mu \ge \mu_1$ at the moment of time $\tau = \tau_0$ the ring stops and the span of the shell has a plastic zone and is still in motion.

The corresponding solution is established just as the solution [1] in the case of motionless supports and with a plastic domain, only instead of initial conditions at $\tau = 1$ as in [1] it is necessary to use initial conditions at $\tau = \tau_0$ from the solutions (3.8)–(3.10).

In this case the expressions for t_2, m, $\dot{W}$ and y_1 are of the same form as in the case of motionless rings [1]. Thus, the conditions

$$-1 \le m(y,\tau) \le 1, \qquad \dot{W}(y,\tau) \ge 0, \qquad y_1(\tau) \le 1$$

will be satisfied if $\mu_2 \le \mu \le \sqrt{6}$ and $\tau \le \tau_1 = P/P_{1M}$. At the moment of time $\tau = \tau_1$ the moving hinge circle reaches the middle of the span and the shell undergoes a motion, corresponding to the regime AB with hinges in the middle and at the end of the span at motionless rings.

In this case the expressions for m and t_2 coincide in the form with the corresponding expressions for the regime AB obtained in [1]. For the velocities of displacement one has

$$\dot{W}(y,\tau) = [\tau_1(P_{1M}-1)-3P_M(\tau-\tau_1)/2]y\ldots$$

$$(\tau_1 \le \tau \le \tau^0, \qquad 0 \le y \le 1, \qquad \mu_1 \le \mu \le \sqrt{6}, \qquad P \ge P_{2*}).$$

The motion of the shell is stopped at moment of time $\tau^0 = P/P_M$ and the maximal residual displacement of the shells with the spans $\mu_1 \le \mu \le \sqrt{6}$ under the loads $P \ge P_{2*}$ is determined by the formula

$$W_1(\tau^0) = P[P(3P_{1M}-P_M)-2P_M P_{1M}]/4P_M P_{1M}. \tag{3.14}$$

4. SHELLS WITH "MODERATE" SPANS UNDER "HIGH" LOADS

The solutions (3.6), (3.7), (3.8) and (3.11) are the same also in the case when $\mu_1 \geq \sqrt{6}$ and $\max(\sqrt{6}, \mu_*) \leq \mu \leq \mu_1$ are the same until the instant of time $\tau = \tau_0$ and at $\tau_0 \leq \tau \leq \tau^0$ the solution would be identical to that for the case of motionless supports and plastic domains near them [1]. But in this case $P \geq P_*$ and the times of motion stopping τ^0 are determined by expression (2.23). The residual displacement is determined by formula (2.24), where

$$W_1(\tau_0) = (\tau_0-\tau_1)[P-\tau_1-(\tau_0-\tau_1)(3P_M-\alpha_1 P_{1*})/4]+[P(P-1)-(P-\tau_1)^2]/2. \quad (4.1)$$

5. SHELLS WITH "LONG" SPANS UNDER "HIGH" LOADS

If $\mu \geq \mu_1 \geq \sqrt{6}$, then at $\tau \geq \tau_0$ the motion of the shell occurs with three plastic regimes

$$AD(0 \leq y \leq u), \qquad AB(u \leq y \leq y_1), \qquad B(y_1 \leq y \leq 1).$$

The corresponding solution is as follows:

$$\begin{aligned} &0 \leq y \leq u: m(y,\tau) = -1, \qquad t_2 = 0, \qquad \dot{W}(y,\tau) = 0 \\ &u \leq y \leq y_1: m = -[4y^3-6(z+2u)y^2+12uy_1y+(u+y_1)(u^2-4uy_1+y_1^2)]/z^3, \\ &\qquad t_2 = -1, \qquad \dot{W}(y,\tau) = (P-\tau)(y-u)/z \\ &y_1 \leq y \leq 1: m(y,\tau) = 1, \qquad t_2 = -1, \qquad \dot{W}(y,\tau) = P-\tau \\ &\qquad (\tau_0 \leq \tau \leq \tau_1, \qquad \mu \geq \mu_1 \geq \sqrt{6}, \qquad P \geq P_{2*}) \end{aligned} \quad (5.1)$$

Here

$$\begin{aligned} z^2 &= (P_{1M}-1)[2+(3-2P_{2*})(P-\tau)^2/(P_{2*}-1)(P-\tau_0)^2], \\ u &= y_0-z+[v\ln(z+v)(y_0-v)/(z-v)(y_0+v)]/4, \\ y_1 &= z+u, \qquad y_0 = [6/\mu^2(P_{2*}-1)]^{\frac{1}{2}}, \qquad v = 2\sqrt{3}/\mu. \end{aligned}$$

The necessary conditions of plasticity [1] will be satisfied, but the condition $\dot{W}_1(\tau) \geq 0$ on the velocity will be satisfied if

$$\begin{aligned} \tau \leq \tau_1 &= P[1-\sqrt{2}(P_{2*}-1)^{\frac{3}{2}}/P_{2*}(v_1 \operatorname{ch}\beta_1+\operatorname{sh}\beta_1)], \\ v_1 &= 2\sqrt{3}/\mu_1, \qquad \beta_1 = (\mu-\mu_1)/\sqrt{3}. \end{aligned} \quad (5.2)$$

At the moment $\tau = \tau_1$ the plastic domain is vanishing in the middle of the span, but the motion continues. At $\tau \geq \tau_1$ the solution in its outward appearance coincides with solution (21) from [1], if we admit that

$$\begin{aligned} u &= \dot{u}_1(\tau-\tau_1)+u(\tau_1), \qquad u(\tau_1) = 1-z_1, \\ \dot{u}_1 &= \dot{u}(\tau_1) = (z_1^2-v^2/2)/z_1(P-\tau_1), \qquad z_1 = z(\tau_1). \end{aligned} \quad (5.3)$$

The time of motion stopping is determined by the expression

$$\tau^0 = \tau_1+z_1(P-\tau_1)/(z_1+v/\sqrt{2}). \quad (5.4)$$

The maximal residual displacement is found from the expression

$$W_1(\tau^0) = a(2z_1 - a\dot{u}_1)/2\dot{u}_1 + (P_{1M} - 1)\dot{u}_1^{-2}\ln(v/z_1\sqrt{2}) + [P(P-1) - (P-\tau_1)^2]/2,$$
$$a = z_1(P-\tau_1)/(z_1 + v/\sqrt{2}) \qquad (\mu \geq \mu_1 \geq \sqrt{6}, P \geq P_{2*}). \tag{5.5}$$

If $\mu \geq \sqrt{6}$, but $\mu_1 \leq \sqrt{6}$, then at $\tau \leq P/2$ the corresponding solution formally coincides with that corresponding to the scheme in Fig. 3a from [1].

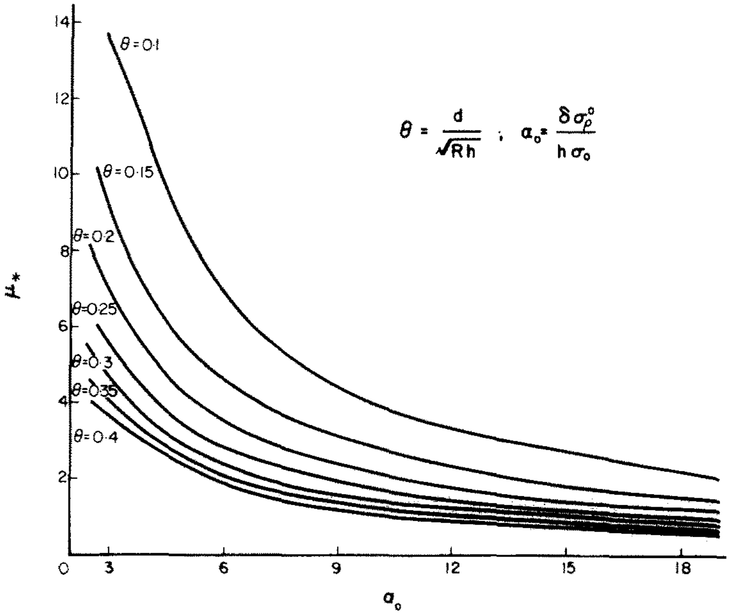

FIG. 3.

At the interval of time $P/2 \leq \tau \leq \tau_1$ the solution coincides in its form with (5.1), if we admit in it

$$z^2 = v^2[1 - 2(1-\tau/p)^2], \qquad u = v/\sqrt{2} - z + [v\ln(z+v)(1-\sqrt{2})/(z-v)(1+\sqrt{2})]/4. \tag{5.6}$$

At the instant of time $\tau = \tau_1$ the plastic domain in the middle of the span vanishes. Then τ_1 is determined from the condition $y_1(\tau_1) = 1$ and has the form

$$\tau_1 = P\{1 - [\sqrt{2}(\sqrt{(2)}\,\mathrm{ch}\,\beta + \mathrm{sh}\,\beta)]^{-1}\}, \qquad \beta = (\mu - \sqrt{6})/\sqrt{3} \tag{5.7}$$

$$z_1 = v(\mathrm{ch}\,\beta + \sqrt{(2)}\,\mathrm{sh}\,\beta)/(\mathrm{sh}\,\beta + \sqrt{(2)}\,\mathrm{ch}\,\beta). \tag{5.8}$$

For $\tau_1 \leq \tau \leq \tau^0$ the solution will outwardly have the same form as the corresponding scheme in Fig. 3a from [1], if we take expressions (5.3) and (5.7) for u, τ_1 and (5.8) for z_1.

The time of stopping τ^0 and the residual displacement are again determined by expressions (5.4) and (5.5), if for τ_1 and z_1 we use in them the formulas (5.7) and (5.8).

6. SHELLS STRENGTHENED BY RIBS IN A SPAN

If in the span between rings a shell is strengthened by a system of closely spaced ring ribs whose rigidity is essentially smaller than that of the supporting rings, then the limiting curve for the span between the supporting rings has the form of the rectangle $A'B'C'D'$ (Fig. 1).

The linear transformation

$$\hat{m} = m, \qquad \hat{t}_2 = t_2 - k_2 \operatorname{sgn} t_2 \tag{6.1}$$

reduces this rectangle to the square $ABCD$.

In (6.1) we introduced the following definitions

$$K_2 = w_2 h_2 \lambda_2, \qquad h_2 = H_2/2H, \qquad \lambda_2 = \sigma_{02}/\sigma_{01}$$

where w_2 is the density of the ribs along the generator, H_2 is their height, σ_{02} is the yield stress of the material in ring ribs, $2H$ and σ_{01} are the thickness and the yield stress of the facings, respectively. Then, if in all the above formulas the value P is replaced by $q = P - K_2$, then we obtain the corresponding solutions for shells, in which the spans between the supporting rings are strengthened by closely spaced ring ribs of "small" rigidity.

With the help of such substitution we obtain from [1] the solution for a shell which has a finite length with absolutely rigid supports and which is strengthened by closely spaced ($\mu < \mu_*$) ring ribs of "small" rigidity.

If the spans of the shell between the supporting rings are strengthened by longitudinal ribs of limited rigidity then a square $ABCD$ (Fig. 1) may be used as a limiting curve for the span if the value

$$m = M_x/M_*$$

is taken as a dimensionless bending moment.

Here M_* is a limiting and bending moment for the shell strengthened by longitudinal ribs and equal to [4], [5]

$$M_* = \sigma_0 H^2(1 + 2h_1 + w_1 h_1^2), \qquad h_1 = H_1/H$$

for a two-layer strengthened shell,

$$M_* = \sigma_0 H^2(1 + 2h_1 + w_1 h_1^2)/(1 + 2h_1)$$

for a one-layer symmetrically strengthened shell, and

$$M_* = \begin{cases} \sigma_0[H^2 + w_1 H_1(H_1 + 2H)/2 - w_1^2 H_1^2/4] \ldots w_1 H_1/2H \leq 1 \\ \sigma_0[w_1 H_1(H_1 + 2H)/2 + w_1 H^2 - (H - w_1 H - w_1 H_1/2)/w_1] \ldots w_1 H_1/2H \geq 1 \end{cases}$$

for a one-layer asymmetrically strengthened shell.

Here $2H$ is the entire thickness of the facing, H_1 is the height of the longitudinal ribs, w_1 is their density in the circumferential direction. Then all the solutions obtained above will be also correct for shells with spans strengthened by longitudinal ribs, if one substitutes c for μ, the value c being determined by the expression

$$c^2 = \mu^2 \sigma_{01} H^2/M_*.$$

If the spans between the supporting rings are strengthened by closely spaced longitudinal and circumferential ribs of "small" rigidity, then to obtain the necessary solutions, it will be sufficient in all of the above formulas to replace P by q and μ by c.

7. CONCLUSION

Depending on the rigidity of the strengthening rings the motion of the shell is different in character. If the load and the parameters of the shell and of the strengthening rings satisfy the inequalities

$$P^0 \le P \le P_0, \qquad \mu \le \mu_*$$

then the shell motion corresponds to axisymmetric squeezing. The value of μ_* determines the boundary of various forms of motion.

If

$$\mu_* \le \mu \le \mu_1 \ldots P_M \le P \le P_{1*}$$

$$\mu \ge \mu_1 \ldots P_M \le P \le P_{1M}$$

then the shell moves in the span between the rings with plastic hinges near the supporting rings and in the middle of the span. The rings remain motionless. The dependence of μ_* and μ_1 on the rigidity a_0 of the ribs at various values of θ is plotted in Figs. 3 and 4.

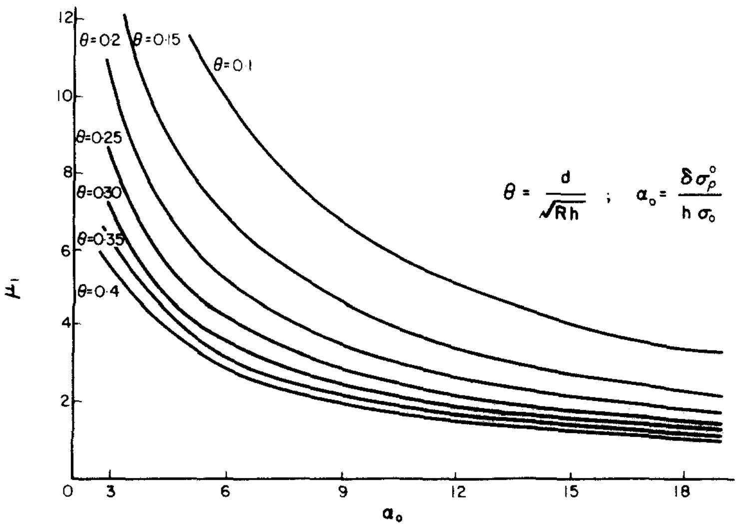

FIG. 4.

If $\mu \ge \mu_1$ and $P_{1M} \le P \le P_{2*}$ the shell also moves in the span and the rings remain motionless but there is a plastic zone. The value P_{1M} determines the load at which a plastic zone appears in case of motion with motionless rings. P_{1*} and P_{2*} are the maximum loads at which the rings remain motionless.

If

$$\mu_* \le \mu \le \min(\sqrt{6}, \mu_1), \qquad P_{1*} \le P \le P_*$$

then the motion of the shell occurs with a hinge in the span and is accompanied by the motion of the rings. The stopping time of the moving rings τ_0 is equal to the ratio of the acting load to that of their load-carrying capacity. At $\mu = \sqrt{6}$ the shell stopping time coincides with the corresponding time [1] in case of motionless rings. P_* is a load at which a plastic zone occurs in the shell with moving rings. It exceeds the corresponding load for the case of absolutely rigid supports.

TABLE 1. SHELLS WITH "LONG" SPANS

	Inequality	Equality	a_0	P_{1*}	P	1·2225	1·4835	1·5335	1·6890	1·8317	2·2122	2·8118	3·5020	4·0000
Rings of limited rigidity	$\mu > \mu_3 > \sqrt{6}$ $P \geq P_{2*}$	$\mu = 6$ $\theta = 0{\cdot}20$	6·0000	1·2225	$W_1(\tau^0)$	0·1335	0·3557	0·4051	0·5771	0·7537	1·3269	2·5271	4·3484	5·9734
			9·0000	1·4835		—	0·3561	0·4067	0·5785	0·7549	1·3292	2·5297	4·3577	5·9815
			12·0000	1·8317		—	—	—	—	0·7578	1·3320	2·5318	4·3617	5·9824
	$\mu > \mu_1 < \sqrt{6}$ $P \geq P_{2*}$	$\mu = 6$ $\theta = 0{\cdot}25$	12·0000	2·2122		—	—	—	—	—	1·3355	2·5384	4·3674	5·9824
			15·0000	2.8118		—	—	—	—	—	—	2·5384	4·3674	5·9824
			18·0000	3·5020		—	—	—	—	—	—	—	4·3674	5·9824
Absolutely rigid rings		$\mu = 6$	$P_{1M} = 1{\cdot}1666$			0·1335	0·3561	0·4064	0·5787	0·7578	1·3355	2·5384	4·3674	5·9824

TABLE 2. SHELLS WITH "SHORT" SPANS

	Inequality	Equality	θ	P_{1*}	P_*	P	2.6917	3·0000	3·3571	3·4812	3·9592	4·0612	5·0000	6·0000
Rings of limited rigidity	$\mu_* < \mu < \mu_2$ $P \geq P_{1*}$	$\mu = 1{\cdot}4$ $a_0 = 15$	0·2500	2·6917	4·4082	$W_1(\tau_0)$	0·6708	1·0735	1·6239	1·8361	2·7552	2·9723	5·1797	8·0588
			0·3000	3·3571	4·2177		—		1·6658	1·8818	2·8171	3·0379	5·2294	8·1303
			0·3500	3·9592	4·0816		—	—	—	—	2.8494	3·0723	5·2411	8·1473
	$P \geq P_{1*}$	$a_0 = 18$	0·2500	3·4812	4·1837			—	—	1·8877	2·8257	3·0472	5·2338	8·1367
Absolutely rigid rings		$\mu = 1{\cdot}4$	$P_M = 2{\cdot}0204$	$P_{1M} = 4{\cdot}0612$			0·6708	1·0909	1·6658	1·8877	2·8494	3·0767	5·2413	8·1475

If $\mu_1 > \sqrt{6}$ then at

$$\max(\sqrt{6}, \mu_*) \leq \mu \leq \mu_1 \quad \text{and} \quad P_{1*} \leq P \leq P_*$$

the motion has the same character, but after the rings stop there occurs a plastic zone in their vicinity. If $P \geq P_{**}$ the shell motion is accompanied by that of the rings and by the occurrence of a plastic zone, the size of the latter decreasing as the rigidity of the rings decreases. In the case under consideration after the load is removed the ring stops before the plastic zone vanishes if $\mu \geq \mu_1$ and after that if $\mu_* \leq \mu \leq \mu_1$.

The calculation done by the given formulae shows that the maximal residual displacement accumulated in the shell during its motion in case of "local" collapse practically does not depend (with an accuracy of about 4 per cent) on ring rigidity and coincides accurately enough with the maximal residual displacement of the smooth clamped shell of the same radius and a length equal to that of the span. True, the distribution of residual displacements in these shells is essentially different.

Thus, in determining the maximal residual displacement there is no necessity to take into consideration the motion of shell supports. Tables 1 and 2 represent some values of maximal residual displacements by way of example (at $\sigma_{01} = \sigma_{02}$, $v_{01} = v_{02}$).

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Абстракт—В работе исследовано динамическое поведение бесконечно-длинной цилиндрической оболочки из идеально жестко-пластического материала, подкрепленной кольцевыми ребрами ограниченной жесткости. Основные предположения идентичны принятым в работе [1], выполненной для оболочек с абсолютно жесткими кольцами.

Определено распределение моментов и прогибов, найдены величины остаточных прогибов и времен окончания движения для различных величин нагрузок, параметров оболочки и подкреиляющих ребер. Показано, что при "местном" разрушении остаточный прогиб, накапливаемый в оболочке за все время движения, практически не зависит от жесткости колец и с хорошей точностью совпадает с максимальным остаточным прогибом оболочки с абсолютно жесткими кольцами. Показывается, что все полученные результаты, путем соответствующих переобозначений, переносятся на случай оболочек, подкрепленных в пролете между опорными кольцами, продольными и кольцевыми ребрами "малой" жесткости. Полученные результаты справедливы также для цилиндрических оболочек конечной длины с подвижными ребрами.